

## THE NONSTATIONARY IDEAL AND THE OTHER $\sigma$ -IDEALS ON $\omega_1$

JINDŘICH ZAPLETAL

ABSTRACT. Under Martin's Maximum every  $\sigma$ -ideal on  $\omega_1$  is a subset of an ideal Rudin-Keisler reducible to a finite Fubini power of the nonstationary ideal restricted to a positive set.

### 0. INTRODUCTION

The purpose of this paper is to show that under a certain set-theoretical hypothesis the  $\sigma$ -ideals on  $\omega_1$  allow of a certain amount of classification. There are certain basic  $\sigma$ -ideals on sets of size  $\aleph_1$ , namely those of the form  $\text{NS}^n \restriction S$ , where  $\text{NS}^n$  is the  $n$ -dimensional Fubini product of the nonstationary ideal on  $\omega_1$ ,  $S \subset \omega_1^n$  is an  $\text{NS}^n$  positive set and  $\text{NS}^0$  is an arbitrary principal ideal on  $\omega_1$ . Let FRI denote this collection of *finite rank* ideals. We prove

**Theorem 1.** *Assume Martin's Maximum [F]. Then every proper  $\sigma$ -ideal on  $\omega_1$  is a subideal of an ideal Rudin-Keisler reducible to a finite rank ideal.*

This theorem suggests the following notion.

**Definition 2.** Given two ideals  $\mathfrak{K}$  and  $\mathfrak{J}$  on their respective underlying sets  $\text{dom}(\mathfrak{K})$  and  $\text{dom}(\mathfrak{J})$  and a function  $f : \text{dom}(\mathfrak{K}) \rightarrow \text{dom}(\mathfrak{J})$ , we say that  $f$  is a *half-reduction* of  $\mathfrak{J}$  to  $\mathfrak{K}$  if for all sets  $X \subset \text{dom}(\mathfrak{J})$ ,  $X \in \mathfrak{J} \rightarrow f^{-1}X \in \mathfrak{K}$ ; and if such a half-reduction exists then  $\mathfrak{J}$  is a *half-reduct* of  $\mathfrak{K}$ . Half-reducts of finite rank ideals will be called *short*; all the other ideals are *long*.

Thus, compared with the Rudin-Keisler reductions, the half-reductions keep only one direction of the equivalence  $X \in \mathfrak{J} \leftrightarrow f^{-1}X \in \mathfrak{K}$ . The relation of half-reducibility is a quasiordering on the class of all ideals coarser than the Rudin-Keisler order, and Theorem 1 states that under Martin's Maximum all  $\sigma$ -ideals on  $\omega_1$  are short. Note that half-reductions involve greater loss of information than the Rudin-Keisler reductions, and the half-reduct generally cannot be recovered from the other ideal and the half-reducing function. For example, the ideal of finite subsets of  $\omega_1$  is a half-reduct of the ideal of countable subsets of  $\omega_1$ , which in turn is a half-reduct of the nonstationary ideal, all witnessed by the identity function on  $\omega_1$ .

Another notion prominent in this paper is

**Definition 3.** Let  $\mathfrak{J}$  be a proper  $\sigma$ -ideal on  $\omega_1$  and let  $P$  be a forcing notion. We say that  $P$  *collapses* the ideal  $\mathfrak{J}$  if the  $\sigma$ -ideal generated by  $\mathfrak{J}$  in the generic extension

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is improper, that is, if  $P$  adds an  $\omega$ -sequence of elements of  $\mathfrak{I}$  whose union is the whole  $\omega_1$ .

A whole new class of stationary-preserving forcing notions collapsing long  $\sigma$ -ideals on  $\omega_1$  will be constructed in Section 1, leading to the proof of Theorem 1. It has been an open problem for a long time whether such forcings exist [B], [Z]. The forcings defined below have certain unusual properties—for example they can increase the boldface projective ordinal  $\delta_2^1$ .

The notation used in this paper follows the set theoretic standard as set forth in [J]. For an ideal  $\mathfrak{I}$  the expression  $\text{dom}(\mathfrak{I})$  stands for the set on which  $\mathfrak{I}$  lives, literally  $\text{dom}(\mathfrak{I}) = \bigcup \mathfrak{I}$  for nonprincipal ideals, and if  $X \subset \text{dom}(\mathfrak{I})$  is an  $\mathfrak{I}$ -positive set (that is a non-element of  $\mathfrak{I}$ ) then the expression  $\mathfrak{I} \restriction X$  stands for the ideal  $\{Y \subset \text{dom}(\mathfrak{I}) : Y \cap X \in \mathfrak{I}\}$ . A Rudin-Keisler reduction of an ideal  $\mathfrak{I}$  to an ideal  $\mathfrak{K}$  is a function  $f : \text{dom}(\mathfrak{K}) \rightarrow \text{dom}(\mathfrak{I})$  such that for every set  $X \subset \text{dom}(\mathfrak{I})$ ,  $X \in \mathfrak{I} \leftrightarrow f^{-1}X \in \mathfrak{K}$ , and if such a function exists then  $\mathfrak{I}$  is called a Rudin-Keisler reduct of  $\mathfrak{K}$ . For a tree  $T$  and a node  $t \in T$  the expression  $T \restriction t$  stands for the tree consisting of all  $T$ -nodes comparable with  $t$ . A trunk of a tree  $T$  is the maximal node of  $T$  comparable with every other node. A splitnode of a tree is any node with at least two distinct immediate successors. If  $T$  is an  $\omega_1$ -tree, then  $T_\alpha$  is the  $\alpha$ -th level of  $T$  and  $T_{<\alpha}$  is the collection of elements of  $T$  below the  $\alpha$ -th level. The partial order  $\text{Power}(\omega_1)$  modulo NS is the factor of the poset  $\text{Power}(\omega_1)$ ,  $\subset$  using the following equivalence on subsets of  $\omega_1$ :  $X \equiv Y$  if  $X \Delta Y$  is nonstationary.

## 1. PROOF OF THEOREM 1

The crux of the argument for Theorem 1 is

**Theorem 4.** *Suppose the nonstationary ideal on  $\omega_1$  is precipitous and  $\mathfrak{I}$  is a long  $\sigma$ -ideal on  $\omega_1$ . Then there exists a stationary-preserving poset  $P(\mathfrak{I})$  which collapses the ideal  $\mathfrak{I}$ .*

Theorem 1 easily follows. Under Martin's Maximum the nonstationary ideal is precipitous—it is even saturated. Suppose  $\mathfrak{I}$  is a long  $\sigma$ -ideal on  $\omega_1$ . Then  $P(\mathfrak{I})$  preserves stationary sets, and for any  $\aleph_1$  many open dense subsets of  $P$  there is a filter meeting all of them. Let  $\{\dot{X}_n : n \in \omega\}$  be a  $P(\mathfrak{I})$ -name such that  $P(\mathfrak{I}) \Vdash \{\dot{X}_n : n \in \omega\} \subset \check{\mathfrak{I}}$  and  $\bigcup_n \dot{X}_n = \check{\omega}_1$ . Consider the sets  $D_n = \{p \in P(\mathfrak{I}) : \exists Y \in \mathfrak{I} \ p \Vdash \dot{Y} = \dot{X}_n\}$  for every number  $n \in \omega$  and  $E_\alpha = \{p \in P(\mathfrak{I}) : \exists n \in \omega \ p \Vdash \check{\alpha} \in \dot{X}_n\}$  for every ordinal  $\alpha \in \omega_1$ . These sets are open dense in  $P(\mathfrak{I})$ , and if  $G \subset P(\mathfrak{I})$  is a filter meeting all of them and  $Y_n$  for  $n \in \omega$  are sets in the ideal such that  $\forall n \in \omega \exists p \in G \ p \Vdash \dot{Y}_n = \dot{X}_n$ , then  $\bigcup_n Y_n = \omega_1$ , proving that  $\mathfrak{I}$  cannot be a proper  $\sigma$ -ideal.

*Proof of Theorem 4.* The forcing  $P(\mathfrak{I})$  consists of all nonempty trees  $T$  of height  $\omega$  such that

- (1)  $T$  is a tree of finite sequences of elements of  $\mathfrak{I}$ ,
- (2) every node  $t \in T$  can be extended to a splitnode  $s \supset t$  of  $T$ ,
- (3) for every splitnode  $s \in T$ , every ideal  $\mathfrak{K} \in \text{FRI}$  and every function  $f : \text{dom}(\mathfrak{K}) \rightarrow \omega_1$ , there is an  $X \in \mathfrak{I}$  such that  $f^{-1}X \notin \mathfrak{K}$  and  $s \frown \langle X \rangle \in T$ .

The ordering is by inclusion. The forcing is designed to add an  $\omega$ -sequence of sets in the ideal  $\mathfrak{I}$ ; the initial segments of this sequence are exactly the trunks of the trees in the generic filter.

It is essentially immediate that the union of the sets on the generic sequence is the whole  $\omega_1$ . Note that for any tree  $T \in P(\mathfrak{I})$ , any splitnode  $s \in T$  and any ordinal  $\alpha \in \omega_1$ , there is a set  $X \in \mathfrak{I}$  such that  $\alpha \in X$  and  $s^\frown \langle X \rangle \in T$ . To see this, apply (3) of the definition of the forcing to the tree  $T$  and node  $s$  with the principal ideal  $\mathfrak{K}$  on  $\omega_1$  based on the ordinal  $\alpha$  and the function  $f = id$ . It is the preservation of stationary subsets of  $\omega_1$  by the poset  $P(\mathfrak{I})$  which is hard to prove; suppose from now on that the nonstationary ideal is precipitous.

Suppose  $S \subset \omega_1$  is a stationary set,  $T$  is a tree in the forcing  $P(\mathfrak{I})$  and  $\dot{C}$  is a  $P(\mathfrak{I})$ -name for a closed unbounded subset of  $\omega_1$ . An ordinal  $\delta \in S$  and a condition  $T' \subset T$  must be found such that  $T' \Vdash \check{\delta} \in \dot{C}$ .

For each ordinal  $\alpha \in S$  fix a one-to-one enumeration  $\alpha = \{\alpha^0, \alpha^1, \alpha^2, \dots\}$  and define an infinite game  $G_\alpha$  between Yossarian and Colonel Cathcart as follows:

$$\begin{array}{rcll} \text{Cathcart} & \mathfrak{K}_0, f_0 & \mathfrak{K}_1, f_1 & \dots \\ \text{Yossarian} & T_0 & T_1 & \dots \end{array}$$

with the following rules:

- (4)  $\mathfrak{K}_n \in \text{FRI}$ ,  $f_n : \text{dom}(\mathfrak{K}_n) \rightarrow \omega_1$ .
- (5)  $T = T_{-1} \supset T_0 \supset T_1 \supset \dots$  is a descending sequence of trees in the poset  $P(\mathfrak{I})$  with strictly increasing respective trunks  $t_n \in T_n$ , and, for each  $n \in \omega$  writing  $X_n$  for the unique set in  $\mathfrak{I}$  such that  $t_{n-1}^\frown X_n \subset t_n$ , we have  $f_n^{-1} X_n \notin \mathfrak{K}_n$ .
- (6) For all numbers  $n$ ,  $T_n \Vdash$  the  $\check{\alpha}^n$ -th element of  $\dot{C}$  is less than  $\check{\alpha}$ .

Yossarian wins the run of the game if he can pass all the infinitely many rounds in accordance with the above rules. The game is closed for Yossarian and therefore determined.

**Lemma 5.** *There is an ordinal  $\delta \in S$  such that Yossarian has a winning strategy in the game  $G_\delta$ .*

*Proof.* Suppose for a contradiction that Colonel Cathcart has a winning strategy  $\sigma_\alpha$  for the game  $G_\alpha$ , for every ordinal  $\alpha \in S$ . Roughly, we shall show that if  $H \subset \text{Power}(\omega_1)$  modulo NS is a generic filter containing the set  $S$  and  $j : V \rightarrow M$  is the associated generic ultrapower, then the strategy  $\sigma_{\omega_1^Y}$  is not winning in the game  $G_{\omega_1^Y}$  in  $V[H]$ . By wellfoundedness of the model  $M$ , this strategy fails in  $M$  too, and by elementarity there must be  $\alpha \in S$  such that the strategy  $\sigma_\alpha$  fails in  $V$ .

Build a sequence  $\langle \mathfrak{A}_n : n \in \omega \rangle$  of maximal antichains of stationary subsets of  $\omega_1$  below  $S$  and a function  $\Theta$  such that the following conditions are satisfied:

- (7)  $\mathfrak{A}_{n+1}$  is finer than  $\mathfrak{A}_n$ , that is, for every set  $B \in \mathfrak{A}_{n+1}$  there is a unique  $A \in \mathfrak{A}_n$  such that  $B \subset A$  modulo the nonstationary ideal. Moreover, for every set  $A \in \mathfrak{A}_n$  and ordinals  $\alpha, \beta \in A$  we have  $\alpha^0 = \beta^0, \dots, \alpha^n = \beta^n$ .
- (8)  $\Theta$  is a function with domain  $\bigcup_n \mathfrak{A}_n$  and range included in  $P(\mathfrak{I})$ .
- (9) For every set  $A \in \mathfrak{A}_n$ , fix the unique sequence  $A_0, A_1, \dots, A_n$  such that  $A_i \in \mathfrak{A}_i$  and  $A = A_n \subset \dots \subset A_1 \subset A_0$ , where the inclusions are taken modulo the nonstationary ideal, and let  $\Theta(A_i) = T_i$ . Then for every ordinal  $\alpha \in A$  the partial play of the game  $G_\alpha$  of length  $n+1$  where Colonel Cathcart follows his strategy  $\sigma_\alpha$  and Yossarian answers with  $T_i$  in turn observes the rules of  $G_\alpha$ ; that is, Yossarian did not forfeit the game so far. In other words, the trees  $T_i, i \leq n$ , are a partial Yossarian's counterplay against all strategies in the set  $\{\sigma_\alpha : \alpha \in A\}$  simultaneously.

Once this is done, the precipitousness of the nonstationary ideal below the set  $S \subset \omega_1$  is applied to get the desired contradiction. Namely, it provides for the existence of an ordinal  $\delta \in S$  and sets  $A_n \in \mathfrak{A}_n$  for every number  $n \in \omega$  with  $A_0 \supset A_1 \supset \dots$ , where the inclusions are taken modulo the nonstationary ideal and  $\delta \in \bigcap_n A_n$ . Look at the play of the game  $G_\delta$  where Colonel Cathcart follows his strategy  $\sigma_\delta$  and Yossarian answers with  $\Theta(A_n)$  in turn. Item (9) above implies that the rules of  $G_\delta$  are observed and Yossarian wins this play, contradicting the choice of the strategy  $\sigma_\delta$ .

Now the antichains  $\mathfrak{A}_n$  are constructed by induction on  $n$ , together with the function  $\Theta$  restricted to  $\mathfrak{A}_n$ . To start the induction just let  $\mathfrak{A}_{-1} = \{S\}$ . Now suppose  $\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n$  as well as  $\Theta \upharpoonright \bigcup_{i \leq n} \mathfrak{A}_i$  have been defined. In the search for  $\mathfrak{A}_{n+1}$  we call a stationary set  $B \subset S$  good if (10-13) hold, where

- (10) there is a unique set  $A \in \mathfrak{A}_n$  with  $B \subset A$ , and
- (11)  $\alpha^{n+1} = \beta^{n+1}$  for all ordinals  $\alpha, \beta \in B$ .

Write  $A_0, A_1, \dots, A_n$  for the unique sequence of sets such that  $A_i \in \mathfrak{A}_i$  for every  $i \leq n$  and  $A_0 \supset A_1 \supset \dots \supset A_n \supset B$ , where the inclusions are taken modulo the nonstationary ideal, and write  $\Theta(A_i) = T_i$  for every  $i \leq n$ . Let an ideal  $\mathfrak{K}_\alpha = \text{NS}^{m_\alpha} \upharpoonright Y_\alpha$ , for some number  $m_\alpha$  and an  $\text{NS}^{m_\alpha}$  positive set  $Y_\alpha$ , and a function  $f_\alpha : \omega_1^{m_\alpha} \rightarrow \omega_1$  constitute the last move dictated to Colonel Cathcart by his strategy  $\sigma_\alpha$  in the partial play of the game  $G_\alpha$ , where Cathcart follows  $\sigma_\alpha$  and Yossarian answers with the trees on the sequence  $\langle T_i : i \leq n \rangle$  in turn, this for every  $\alpha \in B$ . Then there must be a tree  $T_B \in P(\mathfrak{J})$  such that

- (12)  $T_B$  decides the value of “the  $\check{\alpha}^{n+1}$ -th element of the set  $\dot{C}$ ” to be a particular ordinal below  $\min(B)$ , this for every (some)  $\alpha \in B$ , and
- (13)  $T_B \subset T_n$ , and if  $t, s$  are the respective trunks then  $t$  is properly longer than  $s$  and for the unique set  $X \in \mathfrak{J}$  with  $s \cap X \subset t$  we have  $f_\alpha^{-1}X \notin \mathfrak{K}_\alpha$ , for all  $\alpha \in B$ .

*Claim 6. The set of all good sets is dense in the partial order of all stationary subsets of  $S$  with inclusion.*

*Proof.* Fix a stationary set  $B_0 \subset S$ ; a good stationary set  $B \subset B_0$  will be found. Since  $\mathfrak{A}_n$  is a maximal antichain, there is a set  $A \in \mathfrak{A}_n$  with  $B_1 = B_0 \cap A$  stationary; so  $B_1$  satisfies (10). Then use the Fodor lemma to get a stationary set  $B_2 \subset B_1$  satisfying (11). An application of the countable completeness of the nonstationary ideal will yield a stationary set  $B_3 \subset B_2$  such that the ranks  $m_\alpha$  for ordinals  $\alpha \in B_3$  are all equal to one fixed number  $m$ . Now consider the set  $Y = \{\vec{\alpha} \in \omega_1^{m+1} : \vec{\alpha} = \alpha \hat{\ } \vec{\beta}, \alpha \in B_3, \vec{\beta} \in Y_\alpha\} \subset \omega_1^{m+1}$ . This is an  $\text{NS}^{m+1}$ -positive set; consider the function  $f : \vec{\alpha} = \alpha \hat{\ } \vec{\beta} \mapsto f_\alpha(\vec{\beta})$  and the trunk  $t$  of the tree  $T_n$ . By (3) of the definition of the forcing  $P(\mathfrak{J})$  there is a set  $X \in \mathfrak{J}$  such that  $t \cap X \in T_n$  and  $f^{-1}X \notin \text{NS}^{m+1} \upharpoonright Y$ . The last expression means that there is a stationary set  $B_4 \subset B_3$  such that for all  $\alpha \in B_4$  we have  $f_\alpha^{-1}X \notin \text{NS}^m \upharpoonright Y_\alpha = \mathfrak{K}_\alpha$ .

Find a tree  $T_B \subset T_n \upharpoonright t \cap X$  deciding the value of “the  $\check{\alpha}^{n+1}$ -th element of the club  $\dot{C}$ ” for some (all)  $\alpha \in B_4$  to be some  $\beta \in \omega_1$  and let  $B_5 = B_4 \setminus (\beta + 1)$ . The set  $B = B_5 \subset B_0$  is good, as witnessed by the tree  $T_B$ .  $\square$

Let  $\mathfrak{A}_{n+1}$  be a maximal antichain of good stationary subsets of the set  $S$ . Given a set  $B \in \mathfrak{A}_{n+1}$ , let  $\Theta(B) = T_B$ , where  $T_B$  is any tree witnessing goodness of  $B$  as in (12,13). This completes the inductive construction and the proof of the lemma.  $\square$

Then fix an ordinal  $\delta \in S$  such that Yossarian has a winning strategy  $\sigma$  in the game  $G_\delta$ . A tree  $T' \subset T$  will be found such that it forces the ordinal  $\delta$  into the club  $\dot{C}$ —this will complete the proof of stationary set preservation.

*Claim 7.* *There is a tree  $T' \subset T$  in  $P(\mathfrak{J})$  such that if  $t \in T'$  is a splitnode with  $\{s \subseteq t : s \text{ is a splitnode of } T'\} = n + 1$ , then there is a partial run of the game  $G_\delta$  of  $n$  rounds played according to  $\sigma$  such that  $T' \restriction t$  is a subset of the last tree played in this run by Yossarian.*

*Proof.* By a tree induction on  $T$  it will be decided whether a given node  $s \in T$  is a splitnode of  $T'$  or not. In case the answer is affirmative, an auxiliary partial play  $p_s$  of the game  $G_\delta$  will be produced in which Yossarian will follow his winning strategy and

$$(14) \quad s \subset t \rightarrow p_s \subset p_t,$$

(15) if  $U_s$  is the last tree played by Yossarian in the play  $p_s$ , then  $s$  is the trunk of  $U_s$ .

Note that these conditions imply that  $T' \restriction s \subset U_s$  for every node  $s \in T'$ , as desired. To construct the tree  $T'$ , let the trunk  $t$  of  $T$  be the first splitnode of  $T'$  and  $p_t = 0$ . Suppose  $s \in T$  has been decided to belong to  $T'$  and the auxiliary play  $p_s$  has been defined. Varying Cathcart's challenges on the next move of the game  $G_\delta$  and using the axiom of choice, a set  $A$  consisting of pairs  $\langle u, p_u \rangle$  can be obtained so that:

- (16)  $p_u$  is a one-move extension of the play  $p_s$  where Yossarian follows his strategy  $\sigma$  and on his last move plays a tree whose trunk is the sequence  $u$ . So  $s \subset u, u \in T$ .
- (17) For every finite rank ideal  $\mathfrak{K}$  and every function  $f : \text{dom}(\mathfrak{K}) \rightarrow \omega_1$  there is a pair  $\langle u, p_u \rangle$  such that for the unique set  $X \in \mathfrak{J}$  with  $s \cap X \subset u$  we have  $f^{-1}X \notin \mathfrak{K}$ .
- (18) If  $\langle u, p_u \rangle, \langle u', p_{u'} \rangle$  are two distinct elements of the set  $A$ , then already the sets  $X, X' \in \mathfrak{J}$  with  $s \cap X \subset u, s \cap X' \subset u'$  are distinct.

The next level splitnodes of the tree  $T'$  above  $s$  will then be exactly those sequences  $u$  such that  $\langle u, p_u \rangle \in A$  for some play  $p_u$ . This play is unique by (18) above, and it will be the needed auxiliary play for the node  $u$ .  $\square$

Now obviously if  $T' \subset T$  is as in the claim, it must be the case that  $T' \Vdash \check{\delta} \in \dot{C}$ . For if it were not so, there would be a tree  $T'' \subset T'$  and a number  $n$  such that  $T'' \Vdash$  the  $\check{\delta}^n$ -th element of  $\dot{C}$  is bigger than  $\check{\delta}$ . Pick a splitnode  $t \in T''$  such that the set  $\{s \subseteq t : s \text{ is a splitnode of } T'\}$  has size  $m + 1 > n$ . Then there is a partial run of the game  $G_\delta$  of length  $m$  played according to the strategy  $\sigma$  such that the last tree  $U$  played by Yossarian is a superset of  $T' \restriction t$ . Now since  $\sigma$  is winning for Yossarian, it must be the case that  $U \Vdash$  the  $\check{\delta}^n$ -th element of the club  $\dot{C}$  is below  $\check{\delta}$ . As  $T'' \restriction t \subset T''$  and  $T'' \restriction t \subset T' \restriction t \subset U$ , the tree  $T'' \restriction t$  forces the  $\delta^n$ -th element of  $\dot{C}$  to be both smaller than and greater than  $\delta$ , a contradiction.

This completes the proof of stationary set preservation and of Theorem 4.  $\square$

The precipitousness assumption in the above proof seems somewhat out of place. One can ask:

**Question 8.** Is it true in ZFC that for every long ideal on  $\omega_1$  there is a stationary-preserving forcing adding an  $\omega$ -sequence of elements of the ideal whose union is the whole  $\omega_1$ ?

The answer should be negative on the following account. For any ordinal  $\lambda \geq \omega_1$  consider the ideal  $\mathfrak{I}_\lambda = \{X \subset \omega_1 : \text{there is a function } f : \lambda^{<\omega} \rightarrow \lambda \text{ such that no set closed under } f \text{ has ordertype in } X\}$ . These are proper  $\sigma$ -ideals on  $\omega_1$  which are not collapsed in any  $\omega_1$ -preserving extension, since  $\mathfrak{I}_\lambda^V \subset \mathfrak{I}_\lambda^{V[G]}$  in such extensions. Now if the nonstationary ideal is precipitous, these ideals are short, but it seems that in  $L$ -like models the ideal  $\mathfrak{I}_{\omega_2}$  should be long.

## 2. THE SPECIAL CASE OF THE NORMAL IDEALS

The normal ideals on  $\omega_1$  are much easier to treat than the general  $\sigma$ -ideals, partly due to the following phenomenon:

**Lemma 9.** *Suppose  $\mathfrak{I}$  is a normal ideal on  $\omega_1$ . Then the ideal  $\mathfrak{I}$  is long if and only if for no stationary set  $S \subset \omega_1$  do we have  $\mathfrak{I} \subset \text{NS} \restriction S$ .*

*Proof.* The left-to-right implication is trivial: if  $\mathfrak{I} \subset \text{NS} \restriction S$  for some stationary set  $S \subset \omega_1$ , then  $\mathfrak{I}$  is in fact a half-reduct of the ideal  $\text{NS} \restriction S$  as witnessed by the identity function on  $\omega_1$ .

There is a straightforward direct proof of the right-to-left implication—essentially by induction on  $n \in \omega$  one proves that  $\mathfrak{I}$  is not a half-reduct of any ideal of the form  $\text{NS}^n \restriction S$  for any  $\text{NS}^n$ -positive set  $S$ . Instead I offer a rather inefficient proof using Theorem 12 below. If  $\mathfrak{I} \not\subset \text{NS} \restriction S$  for any stationary set  $S \subset \omega_1$ , then there is a stationary-preserving forcing  $P$  collapsing the ideal  $\mathfrak{I}$ ,  $P \Vdash \{\dot{X}_n : n \in \omega\} \subset \tilde{\mathfrak{I}}$  and  $\bigcup_n \dot{X}_n = \tilde{\omega}_1$  for some collection of sets  $\dot{X}_n$ . Now if a function  $f$  were a half-reduction of the ideal  $\mathfrak{I}$  to an ideal  $\text{NS}^n \restriction S$  for some  $\text{NS}^n$ -positive set  $S$ , then  $P$  would collapse that ideal too, using the collection  $\{\check{f}^{-1}\dot{X}_n : n \in \omega\}$ . But this is absurd, since  $P$  preserves stationary sets and a posteriori the positive sets in all Fubini products of the nonstationary ideal.  $\square$

**Corollary 10.** *Suppose that every proper  $\sigma$ -ideal on  $\omega_1$  is short. Then the nonstationary ideal is saturated.*

*Proof.* Suppose that  $\{S_i : i \in I\}$  is a maximal antichain of stationary sets of size greater than  $\aleph_1$ . Then the ideal  $\mathfrak{I} = \{T \subset \omega_1 : |\{i \in I : T \cap S_i \text{ is stationary}\}| \leq \aleph_1\}$  is a proper normal ideal satisfying the assumptions of the previous lemma—so it is a proper long  $\sigma$  ideal on  $\omega_1$ .  $\square$

**Question 11.** Suppose that the nonstationary ideal is saturated. Does it follow that every proper  $\sigma$ -ideal on  $\omega_1$  is short?

The answer should be negative. Another somewhat vague question arising from Corollary 10 is the following. W. Hugh Woodin proved that saturation plus the existence of a measurable cardinal imply the negation of the Continuum Hypothesis. Under the Continuum Hypothesis, is there a combinatorially defined proper long  $\sigma$ -ideal on  $\omega_1$ ?

**Theorem 12.** *Suppose  $\mathfrak{I}$  is a normal ideal on  $\omega_1$  with  $\mathfrak{I} \not\subset \text{NS} \restriction S$  for every stationary set  $S \subset \omega_1$ . Then there is a stationary-preserving forcing  $P_n(\mathfrak{I})$  collapsing the ideal  $\mathfrak{I}$ .*

Compared with the results of Section 1, Theorem 12 shows that the normality of the ideal  $\mathfrak{I}$  makes it possible to get rid of the sometimes awkward assumption of the precipitousness of the nonstationary ideal.

**Theorem 13.** *Suppose  $\mathfrak{I}$  is a normal ideal on  $\omega_1$  such that, for some stationary set  $U \subset \omega_1$ , for no stationary set  $S \subset U$  do we have  $\mathfrak{I} \subset \text{NS} \restriction S$ . Then there is a forcing  $P_n(\mathfrak{I})$  collapsing the ideal  $\mathfrak{I}$  and preserving all the stationary subsets of  $U$ .*

Baumgartner and Taylor [B] asked whether there is a proper  $\sigma$ -ideal  $\mathfrak{I}$  on  $\omega_1$  and an  $\omega_1$ -preserving poset collapsing it. Theorem 13 answers this question in the affirmative: if  $U \subset \omega_1$  is a stationary costationary set, then the ideal  $\mathfrak{I} = \text{NS} \restriction \omega_1 \setminus U$ , and the set  $U$  satisfy the assumption of Theorem 13 and the  $\omega_1$ -preserving forcing  $P_n(\mathfrak{I})$  collapses the ideal  $\mathfrak{I}$ . Note that collapsing this ideal is a much more delicate job than just collapsing the stationarity of the set  $\omega_1 \setminus U$ .

*Proof.* The proofs of Theorems 12, 13 are essentially the same; I will prove Theorem 12 and indicate the changes necessary to get Theorem 13. The elements of the partial ordering  $P_n(\mathfrak{I})$  are nonempty trees  $T$  satisfying the following.

- (19)  $T$  consists of finite sequences of sets in the ideal  $\mathfrak{I}$ .
- (20) Every node  $t \in T$  can be extended to a splitnode  $s \supset t$  of  $T$ .
- (21) If  $s$  is a splitnode of  $T$ ,  $X \in \mathfrak{I}$  is a set and  $\beta \in \omega_1$  is an ordinal, then there is a set  $Y \in \mathfrak{I}$  such that  $\beta \in Y$ ,  $X \subset Y$  modulo a countable set, and  $s \cap Y \in T$ .

The trees are ordered by inclusion. It is immediate that the partial ordering  $P_n(\mathfrak{I})$  adds a countable sequence of sets in  $\mathfrak{I}$ —the sets sitting on the trunks of the trees in the generic filter—whose union is the whole  $\omega_1$ . To prove that the poset preserves stationary subsets of  $\omega_1$  (of  $U$  in the case of Theorem 13), choose a tree  $T \in P_n(\mathfrak{I})$ , a name  $\dot{C}$  for a closed unbounded subset of  $\omega_1$  and a stationary set  $S \subset \omega_1$  ( $S \subset U$  for Theorem 13). A tree  $T' \subset T$  and an ordinal  $\delta \in S$  will be produced such that  $T' \Vdash \check{\delta} \in \check{S} \cap \dot{C}$ , proving the theorems.

For each ordinal  $\alpha \in \omega_1$  fix an enumeration  $\alpha = \{\alpha^n : n \in \omega\}$  and define an infinite game  $G_\alpha$  between Lieutenant Lukáš and Švejk:

Lukáš	$X_0, \beta_0$	$X_1, \beta_1$	...
Švejk	$T_0$	$T_1$	...

with the following rules:

- (22)  $X_n \in \mathfrak{I}, \beta_n \in \omega_1$ .
- (23)  $T = T_{-1} \supset T_0 \supset T_1 \supset \dots$  is a decreasing sequence of trees in the forcing  $P_n(\mathfrak{I})$  with strictly increasing respective trunks  $t_n \in T_n$ , and if  $Y_n \in \mathfrak{I}$  is the unique set such that  $t_{n-1} \cap Y_n \subset t_n$ , then  $\beta_n \in Y_n$  and  $X_n \subset Y_n$  modulo a countable set.
- (24)  $T_n \Vdash$  the  $\check{\alpha}^n$ -th element of the club  $\dot{C}$  is smaller than  $\check{\alpha}$ .

Švejk wins if he passes all the infinitely many rounds observing the above rules. The game is closed for Švejk and therefore determined.

**Lemma 14.** *There is an ordinal  $\delta \in S$  such that Švejk has a winning strategy in the game  $G_\delta$ .*

*Proof.* Suppose for contradiction that Lieutenant Lukáš has a winning strategy  $\sigma_\alpha$  in the game  $G_\alpha$ , for every ordinal  $\alpha \in S$ . By the assumption, there is a stationary set  $R \subset S, R \in \mathfrak{I}$ . Choose a large regular cardinal  $\theta$  and a countable elementary submodel  $M \prec H_\theta$  containing the objects  $\langle \sigma_\alpha : \alpha \in S \rangle, R, \mathfrak{I}, T$  as elements and such that  $\delta = M \cap \omega_1 \in R$ .

By induction on  $n \in \omega$  we shall construct trees  $T_n \in P_n(\mathfrak{I}) \cap M$  such that the sequence  $T_0, T_1, \dots$  is Švejk's counterplay against the strategy  $\sigma_\delta$  observing the rules of the game  $G_\delta$ . This will give a contradiction with the choice of the strategy  $\sigma_\delta$ .

Now suppose the trees  $T_0, \dots, T_n \in M$  were constructed so that Švejk has not yet forfeited his game against the strategy  $\sigma_\delta$  playing these trees. Let  $B = \{\alpha \in S : \text{Švejk has not yet forfeited his game against the strategy } \sigma_\alpha \text{ playing the trees } T_0, \dots, T_n\}$ ; so  $\delta \in B$ . For every ordinal  $\alpha \in B$  let  $X_\alpha \in \mathfrak{I}, \beta_\alpha \in \omega_1$  be Lieutenant Lukáš's challenge in the next round of the game  $G_\alpha$  dictated to him by the strategies  $\sigma_\alpha$ . Let  $X \subset \omega_1$  be the diagonal union of the sequence  $\langle X_\alpha : \alpha \in B \rangle$  and let  $f$  be the function sending  $\alpha$  to  $\beta_\alpha$ . Thus  $B, X, f \in M$  and  $X \in \mathfrak{I}$ . There must be a set  $Y \in \mathfrak{I} \cap M$  such that  $X_\delta \subset X \subset Y$  modulo countable sets,  $f(\delta) \in Y$ , and if  $t$  denotes the trunk of the tree  $T_n$  then  $t \cap Y \in T_n$ , as explained in the following three cases:

- (25) If  $f(\delta) \in \delta$  then  $f(\delta) \in M$ , and the set  $Y \in \mathfrak{I} \cap M$  can be obtained by an application of (21) to  $t, X$  and  $\beta = f(\delta)$  within the model  $M$ .
- (26) If  $f(\delta) = \delta$ , then by an application of (21) within the model  $M$  one can find a set  $Y \in \mathfrak{I} \cap M$  such that  $t \cap Y \in T$  and  $X \cup R \subset Y$  modulo a countable set. But the last inclusion means that the ordinal  $\delta = f(\delta) \in R$  belongs to the set  $Y$  as well, and so  $Y$  is as desired.
- (27) If  $\delta \in f(\delta)$ , let first  $A = \{\alpha \in B : \alpha \in f(\alpha)\}$ . Then the set  $f''A \in M$  is nonstationary and so belongs to the ideal  $\mathfrak{I}$ . By an application of (21), in the model  $M$  one can find a set  $Y \in \mathfrak{I} \cap M$  such that  $t \cap Y \in T$  and  $X \cup f''A \subset Y$  modulo a countable set. But the last inclusion means that the ordinal  $f(\delta) \in f''A$  belongs to the set  $Y$  as well, and so  $Y$  is again as desired.

The tree  $T_{n+1}$  is now obtained by strengthening the tree  $T_n \restriction t \cap Y$  to decide the value of the " $\delta^n$ -th element of the club  $\dot{C}$ " within the model  $M$ . (Note that  $\delta^n \in \delta \subset M$ .) Obviously the value thus forced by the tree  $T_{n+1} \in M$  must belong to the model  $M$  and is therefore smaller than  $\delta$ . So, if Švejk answers with the tree  $T_{n+1}$  to Lukáš's challenge  $X_\delta, \beta_\delta$ , he does not break any rules of the game  $G_\delta$ , and the induction step is complete.  $\square$

The rest of the proof can be literally copied from the argument for Theorem 4.  $\square$

### 3. TECHNICAL PROPERTIES OF THE FORCINGS

With an advent of a new class of partial orders one always asks which preservation properties they share. In our case the most interesting question of this sort is whether the forcings  $P(\mathfrak{I})$  add reals, and what kinds of reals are added.

**Lemma 15.** *Let  $\mathfrak{I}$  be a proper  $\sigma$ -ideal on  $\omega_1$ . Then the forcing  $P(\mathfrak{I})$  adds reals.*

*Proof.* If  $P(\mathfrak{I})$  collapses  $\aleph_1$  this is obvious, so let me assume that  $\aleph_1$  is preserved. Let  $\langle X_n : n \in \omega \rangle$  be a  $P(\mathfrak{I})$ -generic sequence of sets in the ideal  $\mathfrak{I}$  and let  $f : \omega_1 \rightarrow \omega$  be a function defined by  $f(\alpha) =$  the least number  $n$  with  $\alpha \in X_n$ . The range of  $f$  is unbounded in  $\omega$ —otherwise the whole  $\omega_1$  is a union of finitely many sets on the  $X_n$  sequence, and  $\mathfrak{I}$  would not be a proper ideal in the ground model. Let  $\beta \in \omega_1$  be an ordinal such that  $f \restriction \beta$  has unbounded range. I claim that  $f \restriction \beta$  is not in the ground model.



Suppose otherwise; then there must be a function  $g : \beta \rightarrow \omega$  in the ground model with unbounded range and a tree  $T \in P(\mathfrak{J})$  such that  $T \Vdash \check{g} = \check{f} \restriction \check{\beta}$ . Let  $t$  be the trunk of  $T$  of length  $n \in \omega$  and let  $\alpha \in \beta$  be an ordinal such that  $n \in g(\alpha)$ . There is a set  $X \in I$  such that  $t \cap X \in T$  and  $\alpha \in X$ . Obviously  $T \restriction t \cap X \Vdash \check{n} \geq \check{f}(\alpha) \neq \check{g}(\alpha)$ , a contradiction.  $\square$

The argument for the previous lemma may seem rather unrelated to the job the forcings  $P(\mathfrak{J})$  are supposed to perform—that is, collapsing the ideal  $\mathfrak{J}$ . The following two lemmas are a commentary on this.

**Lemma 16.** *It is consistent with the Continuum Hypothesis that there is a stationary costationary set  $S \subset \omega_1$  such that in any forcing extension collapsing the ideal  $\text{NS} \restriction S$  there are new reals.*

*Proof.* Suppose  $\text{CH} + (S \subset \omega_1 \text{ is a stationary costationary set}) + (\text{the ideal } \text{NS} \restriction \omega_1 \setminus S \text{ is saturated}) + (\text{there is a class of measurable cardinals})$ . These assumptions are consistent with a supercompact cardinal [F]. Now suppose  $V[G]$  is a set generic extension of the universe  $V$  such that the ideal  $(\text{NS} \restriction S)^V$  is collapsed in  $V[G]$ , i.e. there is a collection  $\{C_n : n \in \omega\} \in V[G]$  of ground model clubs with  $\bigcap_n C_n \subset \omega_1 \setminus S$ .

Choose a measurable cardinal  $\kappa$  bigger than the poset effecting the extension and a regular cardinal  $\theta > \kappa$ . Choose a countable elementary submodel  $M \prec H_\theta^V$  in  $V[G]$  which contains the club  $C_n$  for every number  $n \in \omega$  as well as  $\kappa$  and the set  $S$ . I claim that the transitive collapse  $\bar{M}$  of the model  $M$  is not in the ground model, proving the lemma. To see this, observe the following.

- (28) The model  $\bar{M}$  is iterable with respect to the ideal  $\overline{\text{NS} \restriction \omega_1 \setminus S}$ . See [W], [X] for the definition and a proof. Here the measurability of the cardinal  $\kappa$  is used.
- (29) By induction on  $\alpha \in \omega_1 + 1$  define elementary submodels  $M_\alpha$  of  $H_\theta^V$ : let  $M_0 = M$ ,  $M_{\alpha+1} = \{f(M_\alpha \cap \omega_1) : f \in M_\alpha\}$  and  $M_\alpha = \bigcup_\beta M_\beta$  for limit ordinals  $\alpha$ . Then the sequence  $\langle \bar{M}_\alpha : \alpha \in \omega_1 \rangle$  of the transitive collapses of these models is an iteration of  $\bar{M}$  using the ideal  $\overline{\text{NS} \restriction \omega_1 \setminus S}$ . This follows from the fact that for each ordinal  $\alpha \in \omega_1$  we have  $\omega_1^{\bar{M}_\alpha} = \omega_1 \cap M_\alpha \in \bigcap_n C_n \subset \omega_1 \setminus S$  (since the model  $M_\alpha$  contains the clubs  $C_n : n \in \omega$  as points) and from the saturation of the  $\text{NS} \restriction \omega_1 \setminus S$  ideal. Moreover,  $\bar{M}_{\omega_1}$  contains all the reals of  $V$  by the continuum hypothesis in  $V$ .

Now if  $\bar{M}$  were indeed an element of  $V$  it would be countable there (since  $V[G] \models |\bar{M}| = \aleph_0$  and  $\omega_1^V = \omega_1^{V[G]}$ ) and so there would be a real  $r \in V$  coding the model  $\bar{M}$ . However, this real then appears in an iterand of  $\bar{M}$ , namely in  $\bar{M}_{\omega_1}$ , an impossibility for iterable models [W], [X].  $\square$

In fact, the proof of this lemma shows that the partial ordering  $P_n(\text{NS} \restriction S)$  can add reals that increase the boldface projective ordinal  $\delta_2^1$ , even in the context of the continuum hypothesis.

**Lemma 17.** *It is consistent to have a stationary costationary set  $S \subset \omega_1$  and a generic extension of the universe with the same reals in which the ideal  $\text{NS} \restriction S$  is collapsed.*

*Proof.* The argument is similar to that of [Z], and I will only sketch it. Assume the generalized continuum hypothesis. By the  $\aleph_0$ -support product add  $\omega_2$  closed unbounded sets  $\langle C_\alpha : \alpha \in \omega_1 \rangle$  of  $\omega_1$  with countable approximations. Then add a

Namba-generic sequence  $\langle \alpha_i : i \in \omega \rangle$  of ordinals converging to  $\omega_2^V$ , and finally, add a set  $S \subset \omega_1 \setminus \bigcap_i C_{\alpha_i}$  with countable conditions.

The arguments of [Z] can be modified to show that the set  $S$  is generic over  $V[\langle C_\alpha : \alpha \in \omega_2 \rangle]$  for the poset adding a subset of  $\omega_1$  with countable conditions. Thus

$$V[\langle C_\alpha : \alpha \in \omega_2 \rangle][S] \models S \subset \omega_1 \text{ is stationary costationary,}$$

$$V[\langle C_\alpha : \alpha \in \omega_2 \rangle][S][\langle \alpha_i : i \in \omega \rangle] \models \text{the ideal } \text{NS} \restriction S \text{ is collapsed} \\ \text{by the clubs } C_{\alpha_i} : i \in \omega,$$

and

$$\begin{aligned} \mathbb{R} \cap V &= \mathbb{R} \cap V[\langle C_\alpha : \alpha \in \omega_2 \rangle] \\ &= \mathbb{R} \cap V[\langle C_\alpha : \alpha \in \omega_2 \rangle][\langle \alpha_i : i \in \omega \rangle][S] \\ &= \mathbb{R} \cap V[\langle C_\alpha : \alpha \in \omega_2 \rangle][S]. \end{aligned}$$

The second equality here follows from the fact that in the presence of the continuum hypothesis Namba forcing does not add reals. Thus the model

$$V[\langle C_\alpha : \alpha \in \omega_2 \rangle][S][\langle \alpha_i : i \in \omega \rangle]$$

is a generic extension of  $V[\langle C_\alpha : \alpha \in \omega_2 \rangle][S]$  adding no new reals and collapsing the ideal  $\text{NS} \restriction S$ .  $\square$

From the arguments of Section 1 it is possible to extract some limitations on the reals added by the forcings  $P(\mathfrak{I})$ , even though at this stage I do not have any direct applications in mind.

**Lemma 18.** *Let the nonstationary ideal be precipitous, let  $\mathfrak{I}$  be a proper long ideal on  $\omega_1$  and let  $\langle r_\alpha : \alpha \in \omega_1 \rangle$  be a modulo finite increasing unbounded sequence of reals—elements of  ${}^\omega\omega$ . Then  $P(\mathfrak{I}) \Vdash \langle r_\alpha : \alpha \in \omega_1 \rangle$  is still unbounded.*

*Proof.* Suppose  $T \in P(\mathfrak{I})$  and  $\dot{s}$  is a  $P(\mathfrak{I})$ -name for a real. I shall produce a tree  $T' \subset T$  and an ordinal  $\alpha \in \omega_1$  such that  $T' \Vdash$  (for an infinite set of integers  $m$  we have  $\check{r}_\alpha(m) > \dot{s}(m)$ ). This will clearly prove the lemma. Now the construction of the tree  $T'$  is a repetition of the argument from Section 1 with the following largely typographical changes. Replace (6) with

$$(6A) \text{ for every } n \text{ there is } k > n \text{ such that } T_n \Vdash \check{r}_\alpha(\check{k}) > \dot{s}(\check{k}).$$

Replace (12) with

$$(12A) \text{ for some integer } k > n + 1, \text{ for all } \alpha \in B \text{ we have } T_B \Vdash \check{r}_\alpha(\check{k}) > \dot{s}(\check{k}).$$

Moreover, in the construction of  $T_B$  and the set  $B_5 \subset B_4$  in the last paragraph of the proof of Claim 6, proceed as follows. First choose a descending sequence  $T_n \restriction t \cap X \supset U_0 \supset U_1 \supset \dots$  of trees in the forcing  $P(\mathfrak{I})$  such that  $U_k$  decides the value of  $\dot{s}(\check{k})$  to be some particular number  $p(k)$ . The function  $p : \omega \rightarrow \omega$  is not a modulo finite bound on the sequence  $\langle r_\alpha : \alpha \in B_4 \rangle$ , and so by an application of countable completeness of the nonstationary ideal there must be a stationary set  $B_5 \subset B_4$  and a number  $k > n + 1$  such that  $r_\alpha(k) > p(k)$  for all  $\alpha \in B_5$ . Let  $B = B_5$ ; the tree  $T_B = U_k$  witnesses the desired goodness of the set  $B$ .

The rest of the argument is similar to the Section 1 treatment and is left to the reader.  $\square$

A similar argument shows that in the  $P(\mathfrak{I})$  extensions ground model Lusin sets of reals do not become meager.

**Lemma 19.** *Suppose the nonstationary ideal is precipitous,  $\mathfrak{I}$  is a proper long ideal on  $\omega_1$  and  $U$  is a Suslin tree. Then  $P(\mathfrak{I}) \Vdash (\check{U} \text{ is a Suslin tree})$ .*

*Proof.* Suppose  $T \in P(\mathfrak{I})$  and  $T \Vdash (\dot{O} \subset \check{U} \text{ is an open dense set})$ . I will produce a tree  $T' \subset T$  and an ordinal  $\alpha \in \omega_1$  such that  $T' \Vdash \check{U}_\alpha \subset \dot{O}$ ; this will prove the lemma. First fix for each  $\alpha \in \omega_1$  enumerations  $\langle t_\alpha^n : n \in \omega \rangle$  and  $\langle s_\alpha^n : n \in \omega \rangle$  of the sets  $U_{<\alpha}$  and  $U_\alpha$  respectively so that for all  $n \in \omega$  we have  $t_\alpha^n <_U s_\alpha^n$ . Now repeat the argument from Section 1 with the following changes. Replace (6) with

(6B) for every  $n$  there is some  $u \in U$  such that  $t_\alpha^n < u < s_\alpha^n$  and  $T_n \Vdash \check{u} \in \dot{O}$ .

Replace (11) with

(11B)  $t_\alpha^{n+1} = t_\beta^{n+1}$  for all  $\alpha \in B$ .

Replace (12) with

(12B) there is  $u \in U$  such that  $t_\alpha^{n+1} < u < s_\alpha^{n+1}$  for all  $\alpha \in B$ , and  $T_B \Vdash \check{u} \in \dot{O}$ .

Moreover, in the construction of the tree  $T_B$  and the set  $B_5 \subset B_4$  in the last paragraph of the proof of Claim 6, proceed as follows. Let  $u_0 \in U$  be the stable value of  $t_\alpha^{n+1}$  for ordinals  $\alpha \in B_4$ . Now all the nodes  $\{s_\alpha^{n+1} : \alpha \in B_4\}$  of the tree  $U$  are above  $u_0$ , and as  $U$  is c.c.c. there must be a node  $u_1 > u_0$  in  $U$  such that  $u_1 \Vdash_U$  “the set  $\{\alpha \in B_4 : s_\alpha^{n+1} \text{ is in the generic filter}\}$  is stationary”. Choose a tree  $T_B \subset T_n \restriction t \cap X$  and a node  $u \in U$  such that  $u > u_1$  and  $T_B \Vdash \check{u} \in \dot{O}$ . By the choice of the node  $u_1$ , the set  $B_5 = \{\alpha \in B_4 : u < s_\alpha^{n+1}\}$  must be stationary. The set  $B = B_5 \subset B_0$  is good, as desired, as witnessed by the tree  $T_B \in P(\mathfrak{I})$ .

The rest of the argument is left to the reader.  $\square$

Another interesting class of questions arises from the comparison between the saturation of the nonstationary ideal and the assertion that long  $\sigma$ -ideals on  $\omega_1$  do not exist. Under Martin’s Maximum the saturation is preserved under c.c.c. forcings [F] and strategically  $\omega_1 + 1$ -closed forcings [V]. Can such forcings add a long ideal?

**Lemma 20.** *Assume Martin’s Maximum. Then every  $\sigma$ -closed  $\aleph_1$ -distributive poset forces that every proper  $\sigma$ -ideal on  $\omega_1$  is short.*

This should be compared with Lemma 4.7 of [V]. The lemma implies that existence of long  $\sigma$ -ideals is not implied by square and similar principles on  $\omega_2$ . I do not know whether the  $\sigma$ -closedness can be dropped from the assumptions. It is clear from the proof that it can be somewhat weakened to  $\omega + 1$ -strategic closure.

*Proof.* First observe that  $\aleph_1$ -distributive forcings preserve the precipitousness of the nonstationary ideal, since the generic ultrapower of the ordinals depends only on  $\text{Power}(\omega_1)/\text{NS}$  and  ${}^{\omega_1}\text{ON}$ , and these objects stay unchanged in the generic extension by such forcings. Now suppose  $Q$  is an  $\aleph_1$ -distributive poset and  $Q \Vdash \check{\mathfrak{I}}$  is a proper long ideal on  $\omega_1$ . Thus  $Q$  is stationary-preserving,  $Q \Vdash$  “NS is a precipitous ideal”, and  $Q \Vdash \dot{P}(\mathfrak{I})$  is stationary preserving; therefore  $Q * \dot{P}(\mathfrak{I})$  is a stationary-preserving forcing.

Fix a large regular cardinal  $\theta$ . By Martin’s Maximum there is an elementary submodel  $M \prec H_\theta$  containing  $\omega_1$  as a subset and  $Q, \check{\mathfrak{I}}$  as points, and a filter  $G \subset Q * \dot{P}(\mathfrak{I}) \cap M$  which meets every dense subset of  $Q * \dot{P}(\mathfrak{I})$  in  $M$ . Let  $\langle X_n : n \in \omega \rangle$

be the sequence of subsets of  $\omega_1$  determined by the  $\dot{P}(\mathcal{I})$  component of the filter  $G$ . Obviously  $\bigcup_n X_n = \omega_1$ . By the  $\aleph_1$ -distributivity of the forcing  $Q$  there must be a descending chain  $\langle q_n : n \in \omega \rangle$  of conditions in  $G \cap Q$  such that  $q_n \Vdash_Q \check{X}_n \in \check{\mathcal{I}}$ . By the  $\sigma$ -closedness of the forcing  $Q$  there is a lower bound  $q \in Q$  of this chain; then  $q \Vdash_Q \{\check{X}_n : n \in \omega\} \subset \check{\mathcal{I}}$  and  $\mathcal{I}$  is not a proper  $\sigma$ -ideal, proving the lemma.  $\square$

**Lemma 21.** *Assume Martin's Maximum. Then every c.c.c. poset of size  $\aleph_1$  forces every proper  $\sigma$ -ideal on  $\omega_1$  to be short.*

This shows that existence of long  $\sigma$ -ideals is not implied by the existence of Lusin sets, Suslin trees and the like objects obtained by adding  $\aleph_1$  many Cohen reals. I do not know whether the size limitation on the c.c.c. poset can be dropped.

*Proof.* Two auxiliary claims of independent interest will be used:

*Claim 22.* *Suppose  $\mathcal{I}$  is a proper long  $\sigma$ -ideal on  $\omega_1$  and  $R$  is a c.c.c. forcing. Then  $R \Vdash$  the ideal generated by  $\check{\mathcal{I}}$  is long.*

*Proof.* First note that the c.c.c.-ness of  $R$  implies that the ideal and the  $\sigma$ -ideal generated by  $\mathcal{I}$  in the generic extension coincide and are both proper. Now suppose  $r_0 \in R, r_0 \Vdash \dot{S} \subset \check{\omega}_1^n$  is an  $\text{NS}^n$ -positive set and  $\dot{f} : \check{\omega}_1^n \rightarrow \check{\omega}_1$  is a function. I shall produce a condition  $r_1 \leq r_0$  and a set  $X \in \mathcal{I}$  such that  $r_1 \Vdash \dot{f}^{-1}\dot{X} \cap \dot{S}$  is  $\text{NS}^n$ -positive, and so  $\dot{f}$  is not a half-reduction of the ideal generated by  $\mathcal{I}$  to  $\text{NS}^n \restriction \dot{S}$ . This will prove the claim.

For an  $\text{NS}^n$ -positive set  $T$  of sequences  $\vec{\alpha} \in \omega_1^n$  there is a condition  $p_{\vec{\alpha}} \leq r_0$  such that  $p_{\vec{\alpha}} \Vdash \vec{\alpha} \in \dot{S}$ , and this can be strengthened to some  $q_{\vec{\alpha}} \leq p_{\vec{\alpha}}$  with  $q_{\vec{\alpha}} \Vdash \dot{f}(\vec{\alpha}) = \check{\beta}_{\vec{\alpha}}$  for some specific ordinal  $\beta_{\vec{\alpha}} \in \omega_1$ . Now the map  $g : \vec{\alpha} \mapsto \beta_{\vec{\alpha}}$  is not a half-reduction of the ideal  $\mathcal{I}$  to  $\text{NS}^n \restriction T$  by the assumptions, and so there is a set  $X \in \mathcal{I}$  such that  $U = g^{-1}X \subset T$  is  $\text{NS}^n$ -positive. By the c.c.c.-ness of the forcing  $R$  again, a condition  $r_1 \leq r_0$  can be found such that  $r_1 \Vdash$  (for an  $\text{NS}^n$ -positive set of  $\vec{\alpha} \in \dot{U}$  the condition  $q_{\vec{\alpha}}$  belongs to the generic filter), and so  $\dot{f}^{-1}\dot{X} \cap \dot{S}$  is  $\text{NS}^n$ -positive as desired.  $\square$

*Claim 23.* *Suppose that all of the ideals  $\mathcal{I}_n$  for  $n \in \omega$  on pairwise disjoint sets of size  $\aleph_1$  are long. Then their product  $\prod_n \mathcal{I}_n = \{X \subset \bigcup_n \text{dom}(\mathcal{I}) : \forall n \in \omega \ X \cap \text{dom}(\mathcal{I}_n) \in \mathcal{I}_n\}$  is long as well.*

*Proof.* Suppose  $S \subset \omega_1^m$  is an  $\text{NS}^m$ -positive set and  $f : \omega_1^m \rightarrow \bigcup_n \text{dom}(\mathcal{I}_n)$  is a function. Then there are a positive set  $T \subset S$  and a number  $n$  such that  $f''T \subset \text{dom}(\mathcal{I}_n)$ , and since  $f \restriction T$  is not a half-reduction of the ideal  $\mathcal{I}_n$  to  $\text{NS}^m \restriction T$  there is a set  $X \in \mathcal{I}_n$  with  $f^{-1}X \cap T$   $\text{NS}^m$ -positive. As  $X \in \prod_n \mathcal{I}_n$ , this set also witnesses the fact that  $f$  is not a half-reduction of  $\prod_n \mathcal{I}_n$  to  $\text{NS}^m \restriction S$ . The claim follows.  $\square$

For the proof of the lemma, first note that c.c.c. forcings preserve the precipitousness of the nonstationary ideal, and that under Martin's Maximum c.c.c. is productive. Now suppose  $Q$  is a c.c.c. poset of size  $\aleph_1$ , and  $Q \Vdash (\check{\mathcal{I}} \text{ is a proper long ideal on } \omega_1)$ . Then  $Q^{<\omega} =$  finite support product of the copies  $Q_n$  (for  $n \in \omega$ ) of the poset  $Q$  is c.c.c. as well, and, if  $G \subset Q^{<\omega}$  is a generic filter,  $G_n = G \cap Q_n$  and  $\mathcal{I}_n =$  the ideal in  $V[G]$  generated by  $\check{\mathcal{I}}/G_n$ , then by Claim 22 used in the respective models  $V[G_n]$  the ideals  $\mathcal{I}_n$  are long in  $V[G]$  for all natural numbers  $n$ . By Claim 23, the ideal  $\mathfrak{K}$  on  $\omega_1 \times \omega$  defined by  $A \in \mathfrak{K} \leftrightarrow \forall n \in \omega \ \{\alpha \in \omega_1 : \langle \alpha, n \rangle \in A\} \in \mathcal{I}_n$  is a long ideal in  $V[G]$ . By the results of Section 1, then, in  $V[G]$  the forcing  $P(\mathfrak{K})$

is stationary-preserving, and back in the ground model, the forcing  $Q^{<\omega} * P(\mathfrak{K})$  is stationary-preserving.

Choose a large regular cardinal  $\theta$ . Martin's Maximum implies that there is an elementary submodel  $M \prec H_\theta$  containing  $\omega_1$  as a subset and  $Q, \dot{\mathcal{J}}$  as elements and a filter  $G \subset Q^{<\omega} * P(\mathfrak{K}) \cap M$  meeting all the dense subsets of the forcing  $Q^{<\omega} * P(\mathfrak{K})$  in  $M$ . For every number  $n \in \omega$  let  $G_n = G \cap Q_n$ , and let  $\langle A_m \in \omega \rangle$  be the sequence of sets obtained through the  $P(\mathfrak{K})$  component of the filter  $G$ ; so  $\bigcup_m A_m = \omega_1 \times \omega$  and for every pair  $m, n \in \omega$  of natural numbers the set  $\{\alpha \in \omega_1 : \langle \alpha, n \rangle \in A_m\}$  belongs to the ideal generated by  $\mathcal{I}/G_n$ . By the definitions, for every pair  $m, n$  of natural numbers there is a  $Q$ -name  $\tau_n^m \in M$  so that  $Q \Vdash (\tau_n^m \in \dot{\mathcal{J}} \text{ and } \{\alpha \in \omega_1 : \langle \alpha, n \rangle \in A_m\} \subset \tau_n^m / G_n)$ . I claim that  $Q \Vdash (\bigcup_{m,n} \tau_n^m = \check{\omega}_1 \text{ and } \dot{\mathcal{J}} \text{ is not to be a proper } \sigma\text{-ideal})$ . To see this, fix a condition  $q \in Q$  and an ordinal  $\alpha \in \omega_1$ , and produce a condition  $r \leq q$  and numbers  $n, m$  with  $r \Vdash \check{\alpha} \in \tau_n^m$ . Since  $\bigcup_n G_n = Q$  (this is the only place where the size restriction on  $Q$  is used), there is a number  $n$  with  $q \in G_n$ . By the choice of the names  $\tau_n^m$  there is an  $m$  such that  $\alpha \in \tau_n^m / G_n$ ; so there must be a condition  $r \leq q$  in the filter  $G_n$  with  $r \Vdash \check{\alpha} \in \tau_n^m$ .  $\square$

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DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NEW HAMPSHIRE 03755  
*E-mail address*: zapletal@dartmouth.edu